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On the Structure of a One-Dimensional Quotient Field

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In this paper we shall develop some properties of M-domains, i.e., commutative integral domains whose quotient fields have homological dimensions at most one. Such domains are of interest in the study of divisible modules [5] and of cotorsion modules [6].

Actually the M-domains possessing the nicest properties are those whose quotient fields are countably generated modules. In [3] Kaplansky clarified the situation somewhat by showing that the quotient field of a quasilocal M-domain is a countably generated module. While this result is simply false in the global case (e.g., in the case of a polynomial ring over an uncountable field), in our main result (Corollary 2.4) we are able to obtain a suitable generalization, which is that for an M-domain, the quotient field modulo the ring has the property that any of its countably generated submodules may be enlarged to a countably generated direct summand. This latter condition is a reasonably good substitute for countable generation of the quotient field, and we use it to complete the reasoning in a circle of ideas due to Matlis [5]. This is Theorem 2.6: For an integral domain R the following are equivalent: (i) R is an M-domain; (ii) the torsion submodule of a divisible R-module is always a direct summand; and (iii) each divisible module is the quotient of an injective module.

This note is divided into three sections. In Section 1 we generalize results of Kaplansky to the case of an M-domain. Section 2 contains the main result and its application to divisible modules. In Section 3, we note some general facts about the module $\underset{\sim}{Q}/R$ (R any integral domain, $\underset{\sim}{Q}$ its quotient field) which serve to motivate the construction of direct summands of $\underset{\sim}{Q}/R$ in Section 2. We also give an example which answers a question (unpublished) of Kaplansky; viz., that even if all the localizations of a domain R at its prime ideals are M-domains, R need not be an M-domain.

Most of our terminology and notations are standard and are identical with those explained in the papers of Matlis and Kaplansky [2–7]. Once and for all, we fix the following convention which is in force throughout: R denotes a commutative integral domain with unit element, and $\underset{\sim}{Q}$ stands for the quotient field of R.

1. Preliminaries

We shall have need of several results on M-domains. These results require but slight generalization of the techniques of Kaplansky as set forth in [3], but for completeness we supply proofs. We utilize the theory of homological dimension as given in [4], and we denote the homological dimension of the R-module A by $d_R A$. *Throughout this section we assume that R is an M-domain*: $d_R \underset{\sim}{Q} \leq 1$. Here and in the remainder of this paper, we employ the following terminology:

Definition. A semigroup in R is a multiplicatively closed subset of *nonzero* elements in R which contains 1.

Proposition 1.1. *Any countable semigroup S_0 in R may be enlarged to a countable semigroup S in R with $d_R \underset{\sim}{Q}/R_S \leq 1$.*

Proof. Let F be the free R-module on a base $\{x_t : t \in R \text{ and } t \neq 0\}$. Map F onto $\underset{\sim}{Q}$ by declaring that $x_t \mapsto t^{-1}$, and write K for the kernel of this map. Now we have assumed that K is projective so by [2, Theorem 1] $K = \bigoplus K_i$ (i running through some index set) where each K_i is a countably generated, projective R-module. For any set T of nonzero elements in R, we let $F(T)$ stand for the span of $\{x_t : t \in T\}$, and we let $K(T) = K \cap F(T)$. When T is a countable semigroup, $K(T)$ is countably generated by $\{x_t - ux_{tu} : t, u \in T\}$.

Let S_0 be a countable semigroup in R. As $K(S_0)$ is countably generated, there is a countable sum L_0 of K_i's which contains $K(S_0)$. Then we can find a countable semigroup S_1 so that $L_0 \subset K(S_1)$. Continuing with this procedure, we construct an infinite sequence: $K(S_0) \subset L_0 \subset K(S_1) \subset \cdots \subset L_n \subset K(S_{n+1}) \subset L_{n+1} \subset \cdots$, where each S_n is a countable semigroup in R and each L_n is a countable direct sum of certain K_i's. We shall show that $S = \bigcup S_n$ is a semigroup with the desired properties. Certainly S is countable and contains S_0. Note that $K(S) = \bigcup K(S_n) = \bigcup L_n$ is a direct sum of K_i's and so $K(S)$ is a direct summand of K. We have a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K(S) \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K/K(S) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F(S) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & F/F(S) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R_S \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & Q/R_S \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$F/F(S)$ is free while $K/K(S)$ is seen to be projective. We then read off that $d_R Q/R_S \leqslant 1$, which proves the lemma.

We now proceed to give three results ending in Corollary 1.4 which states the key property of the semigroups S with $d_R Q/R_S \leqslant 1$. For this no countability assumption on S is needed.

PROPOSITION 1.2. *Let A be an R-submodule of Q such that $d_R Q/A \leqslant 1$. Then for any nonzero $x \in R$, A/xA is $R/(x)$-projective.*

Proof. Obviously, as the homological dimensions of Q and Q/A are at most one, we have that $d_R A \leqslant 1$. From [4, Theorem 8, p. 176] it follows that $d_{R/(x)} A/xA$ is either zero or one. Next observe that Q/xA and Q/A are isomorphic, and by assumption their common dimension is at most one. Thus from the exact sequence, $0 \to A/xA \to Q/xA \to Q/A \to 0$, we can conclude that $d_R A/xA \leqslant 1$. With this information we may now rule out the possibility that $d_{R/(x)} A/xA = 1$; were this dimension one, by [4, Theorem 3, p. 172] it would force $d_R A/xA = 2$, which is not the case.

COROLLARY 1.3. *Suppose that R is quasilocal and A is an R-submodule of Q. Then $A = Q$ if it satisfies the following properties: $A \neq 0$; $d_R Q/A \leqslant 1$; and A is divisible by a nonunit of R.*

Proof. As A is nonzero, it suffices to show that $A = xA$ for any nonzero x in R. By Proposition 1.2, A/xA is a projective module over $R/(x)$ and is free by [2, Theorem 2]. But A/xA is divisible by a nonunit of $R/(x)$ and so must be zero.

Remark. At this point one may combine Proposition 1.1 and Corollary 1.3 to obtain Kaplansky's theorem [3] that the quotient field of a quasilocal M-domain is countably generated.

We now define the property in which we are interested. The terminology was suggested by D. Eisenbud (by analogy with sheaves).

DEFINITION. Let $A \subseteq B$ be R-modules (R any commutative ring). We say that A is a restriction of B if for every prime ideal P of R either $A_P = B_P$ or $A_P = 0$. Note that the definition is unchanged with the word *prime ideal* replaced by maximal ideal.

COROLLARY 1.4. *Let R be an M-domain and S be a semigroup in R with $d_R Q/R_S \leqslant 1$. Then R_S/R is a restriction of Q/R.*

Proof. If P is a prime ideal of R and $P \cap S = \varnothing$, then $(R_S)_P = R_P$ whence $(R_S/R)_P = (R_S)_P/R_P = 0$. When $P \cap S \neq \varnothing$, then $A = (R_S)_P$ satisfies the hypotheses of Corollary 1.3 over the quasilocal ring R_P; hence $(R_S)_P = Q$ and $(R_S/R)_P = Q/R_P = (Q/R)_P$.

2. Main Theorem

Theorem 2.1. *Suppose that R is an M-domain and that S is a multiplicative semigroup of nonzero elements in R with $d_R Q/R_S \leqslant 1$. Then R_S/R is a direct summand of Q/R.*

We break the proof of Theorem 2.1 up into two lemmas. First fix S with $d_R Q/R_S \leqslant 1$. Partition the set X of maximal ideals of R into disjoint subsets V and W, where V (standing for void) $= \{M \in X : M \cap S \text{ is void}\}$ and $W = \{M \in X : M \cap S \text{ is not empty}\}$. We also let $A = \bigcap R_M$, $M \in V$ and $B = \bigcap R_M$, $M \in W$.

Lemma 2.2. (i) $R_S = A$.

(ii) *R_S/R is a divisible module.*

(iii) *R_S/xR_S is a cyclic, projective $R/(x)$-module for any nonzero x in R.*

Proof. (i) For any $M \in V$, $R_S \subset R_M$, so $R_S \subset A$. From Corollary 1.4, it is easy to show that upon localization at a maximal ideal of R, R_S and A become equal; hence $R_S = A$.

(ii) By Corollary 1.4, R_S/R is a restriction of Q/R. This implies that upon localization at a maximal ideal of R, R_S/R becomes a divisible R-module. But then R_S/R is itself a divisible R-module.

(iii) Let x be a nonzero element of R. By (ii) $x(R_S/R) = R_S/R$, which can be rewritten as: $R_S = xR_S + R$. This formula clearly displays the fact that R_S/xR_S is cyclic; the statement about projectivity follows by Proposition 1.2.

Owing to (i) of the preceeding lemma, the statement of the following lemma also proves Theorem 2.1.

Lemma 2.3. *$A/R \oplus B/R = Q/R$.*

Proof. Now $A \cap B = \bigcap R_M$ (M a maximal ideal of R) $= R$, so $A/R \cap B/R = 0$. It remains to verify that $A/R + B/R = Q/R$ or, equivalently, $A + B = Q$. Suppose x is neither zero nor a unit of R, and we shall show that x^{-1} is in $A + B$, which will prove the lemma.

By (iii) of Lemma 2.2, we may write $R/(x) = (e) \oplus (f)$, where e and f are orthogonal idempotents and where (f) is isomorphic to R_S/xR_S. For a maximal ideal M in R which contains x, we have: $e \in M/(x) \Leftrightarrow M(f) \neq (f) \Leftrightarrow M(R_S/xR_S) \neq R_S/xR_S \Leftrightarrow MR_S \neq R_S \Leftrightarrow S \cap M = \varnothing$, i.e., $M \in V$. For a maximal ideal M of R which contains x, precisely one of the elements e

or f lies in $M/(x)$. Summing up this discussion, we have that, for any maximal ideal M of R which contains x,

(1) $e \in M/(x) \Leftrightarrow M \in V$.

(2) $f \in M/(x) \Leftrightarrow M \in W$.

We may always choose *nonzero* elements a and b of R so that $a + (x) = e$, $b + (x) = f$, and $a + b = 1$. Notice that $x^{-1} = ax^{-1} + bx^{-1}$ and also that abx^{-1} lies in R. We claim that $ax^{-1} \in A$ and $bx^{-1} \in B$. In effect, to say that $ax^{-1} \in A$ is to say that $ax^{-1} \in R_M$ for every M in V. In case x is not in M, it is obvious that $ax^{-1} \in R_M$. In case $x \in M$, (2) implies that b is not in M. If we rewrite ax^{-1} in the form $(abx^{-1})\, b^{-1}$, it is now clear that ax^{-1} lies in R_M. A symmetric argument demonstrates that bx^{-1} lies in B, and this concludes the proof.

Any countably generated submodule of Q/R is contained in a submodule of the form R_{S_0}/R, where S_0 is a countable semigroup in R. To obtain such an S_0 assemble representatives $a_i b_i^{-1}$, $i = 1, 2, 3, \ldots$ in Q for generators of the submodule and let S_0 be the semigroup of all possible products of 1 and the b_i's. Juxtaposing Theorem 2.1 and Proposition 1.1, we have our main result:

Corollary 2.4. *Let R be an M-domain with quotient field Q. Then any countably generated submodule of Q/R may be enlarged to a direct summand of the form R_S/R, S a countable semigroup of nonzero elements in R.*

We also have an immediate

Corollary 2.5. *If R is an M-domain and Q/R is indecomposable, then Q is a countably generated R-module.*

With Corollary 2.4 at hand, we are able to adapt techniques of Matlis to close the circle of ideas in [5]. For comments on the conditions involved in Theorem 2.6 and some partial results in its direction, consult the remarks in [6; pp. 5, 6, 55].

Definition. Over an integral domain, a module is called h-divisible if it is the quotient of an injective module or, equivalently, if it is a quotient of a direct sum of copies of the quotient field. [Notice that over any domain R (noetherian or not) a direct sum of any number of copies of the quotient field is an injective R-module.] The latter definition shows that any sum of h-divisible submodules of a given module is again an h-divisible submodule.

Theorem 2.6. *For any integral domain R, the following are equivalent properties:*

(i) *R is an M-domain.*

(ii) *The torsion submodule of a divisible module is always a direct summand.*

(iii) *Any divisible module is the quotient of an injective module.*

Proof. In [5, Theorems 1.1 and 1.2], Matlis shows that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Assume that (i) holds. Suppose we are given a divisible module D and wish to show that D is h-divisible. We do this first for T, the torsion submodule of D, which is, of course, also a divisible module.

Let t be any element of T and choose a nonzero $s \in R$ with $st = 0$. Then Corollary 2.4 asserts the existence of a countable semigroup S in R such that $s \in S$ and R_S/R is a direct summand of Q/R. In particular, R_S/R is h-divisible. Let $s = s_1, s_2, \ldots$ be an enumeration of S. Set $t_1 = t$. Choose $t_2 \in T$ so $s_2 t_2 = t_1$. Inductively, complete this to an infinite sequence, $t_1, t_2, \ldots$, where $s_{n+1} t_{n+1} = t_n$. There is a well-defined homomorphism from R_S/R into T which sends each $(s_1 \cdots s_n)^{-1} + R$ to the element t_n. The image of R_S/R in T is an h-divisible submodule which contains t. T is, therefore, the sum of all its h-divisible submodules, which is to say that T itself is h-divisible.

To show that D (as above) is h-divisible, we repeat an argument from [5]. Now D/T is a direct sum of copies of Q (or is the zero module). In any case, $d_R D/T \leqslant 1$. We have shown that there is a short exact sequence, $0 \to A \to I \to T \to 0$, where I is injective. There is an exact sequence, $\mathrm{Ext}_R^1(D/T, I) \to \mathrm{Ext}_R^1(D/T, T) \to \mathrm{Ext}_R^2(D/T, A)$. But the ends of the sequence are zero—the left end as I is injective and the right end as $d_R D/T \leqslant 1$. Therefore, $\mathrm{Ext}_R^1(D/T, T) = 0$ also. Interpreting this Ext as the group of extensions of T by D/T, we conclude that T is a direct summand of D. $D \approx T \oplus D/T$ is h-divisible as T and D/T are known to be h-divisible, and the proof is finished.

3. Remarks

In this concluding section we shall supply a direct argument for a result of Matlis [7, Corollary 4.2] and exhibit two applications of it which are relevant to the study of M-domains. Note that Corollary 2.5 and Lemma 3.1 show how Kaplansky's theorem in the quasilocal case is a special case of Corollary 2.4.

Lemma 3.1 (Matlis). *Let R be a quasilocal domain with quotient field Q. Then the module Q/R is indecomposable.*

Proof. Suppose $Q/R = A \oplus B$ with neither A nor B the zero submodule of Q/R. R is not a field. Therefore, the images in Q/R of the inverses of nonzero elements in the maximal ideal of R generate Q/R as an R-module, and there must be a nonzero x in the maximal ideal of R such that $x^{-1} + R$ has both its A- and B-components nonzero. [If $y^{-1} + R$ has nonzero A-component and $z^{-1} + R$ has nonzero B-component, take $x^{-1} = (yz)^{-1}$.] Each of these components is annihilated by x, from which it follows that the A-component (also the B-component) of $x^{-1} + R$ has the form $ax^{-1} + R$ where a is in R. Now a cannot be a unit as $x^{-1} + R$ is not in A. Therefore, $(1 - a)x^{-1} + R$ is in B, $1 - a$ is a unit, and $x^{-1} + R$ is in B, a contradiction, and the lemma is proved.

We shall now show that the (unique) complement of a direct summand of Q/R where R is any domain, may be constructed in an analogous fashion to that used in Section 2. Lemma 3.1 implies that a direct summand of Q/R is a restriction of Q/R. In more detail, supposing $Q/R = A \oplus B$, we have that, for a prime ideal P, $Q/R_P = A_P \oplus B_P$. So, by the lemma, $\{A_P, B_P\} = \{0, Q/R_P\}$. This observation leads (by an easy localization argument which we omit) to the following information on the summands of Q/R.

PROPOSITION 3.2. *Suppose that R is any integral domain, and $Q/R = A \oplus B$. Let A^* denote the inverse image of A in Q (under its natural projection on Q/R). Then $A^* = \bigcap R_M$, M running over the maximal ideals of R with $A_M = 0$ (equivalently, $M \in$ support B). Consequently, the complement of a direct summand of Q/R is unique.*

Remark. Matlis has proved a stronger result than the last statement of the proposition, viz., that the endomorphism ring of Q/R is commutative [6, Proposition 5.1].

Now let R be an M-domain with quotient field Q. If S is a multiplicative semigroup in R, then R_S is an M-domain. In particular, if P is a prime ideal, R_P is a quasilocal M-domain, so that Q is countably generated over R_P. Thus, if R is an M-domain, it is locally powerful (in the terminology of Matlis [6]). We shall now provide an example to show that the converse is false.

We will first describe a topological space X which is to be the spectrum of our example. Let I be an *uncountable* index set, and let the space X consist of distinct points $\{M_\alpha : \alpha \in I\} \cup \{P\} \cup \{P_\alpha : \alpha \in I\} \cup \{0\}$. Before defining the topology, partially order X by decreeing that the M_α's are the maximal elements of X, and that 0 is the minimal element of X. The only other relations are to be: $0 \leqslant P_\alpha \leqslant M_\alpha$ and $0 \leqslant P \leqslant M_\alpha$ for every α. Consult Fig. 1.

For x in X let $V_x = \{y \in X : x \leqslant y\}$. The collection of the V_x's together with all finite unions of them are precisely the closed sets in the topology we place on X. One verifies that in this topology X is a noetherian space (every open subset is quasicompact) and that the V_x's are precisely the irreducible closed subsets, so, in particular, $X = V_0$ is irreducible. In addition, X is a T_0-space and each irreducible closed subset has a generic point. It is also clear that the topological order on X by $x \leqslant y$ if $y \in$ closure $\{x\}$, is just the ordering described above.

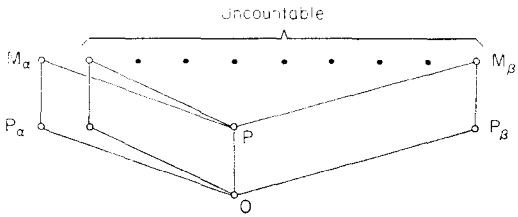

FIGURE 1

Now by the powerful theorem of Hochster [1, Theorem 6], a topological space Y is the spectrum of some commutative ring if and only if: (i) Y is a quasicompact T_0-space, (ii) every nonempty irreducible closed subset of Y has a generic point, and (iii) the collection of quasicompact open sets in Y is closed under finite intersection and forms an open basis for Y. It follows immediately that if such a space is irreducible, it can be realized as the spectrum of an integral domain. It has been remarked already that (i) and (ii) hold for the space X, and (iii) follows at once from the fact that X is a noetherian space. Moreover, as X is irreducible, there is by Hochster's theorem an integral domain R whose spectrum of prime ideals is homeomorphic to X. Before showing that any such domain will provide the desired example, we insert a lemma.

LEMMA 3.3. *Let R be a domain, and let* spec R *denote the prime spectrum of R with the Zariski topology. Then the quotient field Q of R is a countably generated R-module if and only if* $\{0\} \subset$ spec R *is the intersection of a countable number of open sets of* spec R.

Proof. First observe that Q is a countably generated module if and only if there is a countable semigroup S in R such that $R_S = Q$. Now for r in R let $D(r) = \{N \in \text{spec } R : r \notin N\}$. Then the previous condition is equivalent to the existence of a countable set T of nonzero elements of R which meets every nonzero prime ideal of R, i.e., $\{0\} = \bigcap D(t)$, $t \in T$. Now as the collection of all $D(r)$, r in R, is an open basis for the topology on spec R, the last criterion is equivalent to the statement that $\{0\}$ is the intersection of a countable number of open sets of spec R.

Now let R be an integral domain whose spectrum of prime ideals is homeomorphic to X, and let $\underset{\sim}{Q}$ be the quotient field of R. We shall, for notational convenience, identify X with spec R.

LEMMA 3.4. For any prime N of R, $d_{R_N}\underset{\sim}{Q} \leqslant 1$, but $d_R\underset{\sim}{Q} > 1$.

Proof. Let N be a prime of R. By inspection (consult Fig. 1) only a finite number of primes are contained in N; so R_N has a finite number of prime ideals. It is, therefore, possible to choose a nonzero element t which lies in every nonzero prime of R_N and then the powers of t^{-1} provide a countable module generating set for $\underset{\sim}{Q}$ over R_N. It is well-known that this implies $d_{R_N}\underset{\sim}{Q} \leqslant 1$.

The nonzero prime ideal P is contained in every maximal ideal of R, and it follows by an easy localization argument and Lemma 3.1 that $\underset{\sim}{Q}/R$ is an indecomposable module. If $d_R\underset{\sim}{Q} = 1$, Corollary 2.5 would imply that $\underset{\sim}{Q}$ were a countably generated R-module. But we can show $\underset{\sim}{Q}$ is not a countably generated R-module. A glance at the topology on the space X reveals that any nonempty open subset of X contains all but a finite number of the P_α's. This fact makes it clear that the topological condition of Lemma 3.3 does not hold for X; wherefore $\underset{\sim}{Q}$ is not a countably generated R-module. Thus $d_R\underset{\sim}{Q} > 1$.

Remark. If we took the indexing set I to have cardinality $\aleph_1$, then one could conclude that $d_R\underset{\sim}{Q} = 2$.

An open question which is connected with the situation above is whether or not an integral domain of Krull dimension one (and therefore locally powerful) must be an M-domain. In closing we mention one other problem. If R is an M-domain, is $\underset{\sim}{Q}/R$ a DSC-module (direct sum of countable generated modules)? This is true if R is an h-local domain or if R is a UFD. The result, if proved, would characterize M-domains, for the converse implication ($\underset{\sim}{Q}/R$ DSC $\Rightarrow d_R\underset{\sim}{Q} \leqslant 1$) is easy to prove.

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